MINIMALIST PRACTICAL NUMBERS

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ABSTRACT. A natural number n is practical if every smaller number can be written as a sum of distinct divisors of n. We say that a practical number n is minimalist if this representation is unique. In this note, we prove that a practical number is minimalist if and only if it is a power of 2.

1. INTRODUCTION

A natural number n is a practical number if for all natural numbers $m < n$, we may write m as a sum of distinct divisors of n. Srinivasan defined these numbers in 1948 [5], and within a decade Stewart completely characterized the practical numbers in [6] (see also [3]). Since then, practical numbers have been studied both arithmetically [1, 4] and analytically [2, 7].

As usual, we write $\sigma : \mathbb{Z}_{>0} \to \mathbb{Z}$ for the sum-of-divisors function, which is given by

$$\sigma : n \mapsto \sum_{d|n} d.$$

Theorem 1.1 ([6], Theorem 1, Section 3)**.** *Let n be a natural number, and write $n = p_1^{a_1} \dots p_r^{a_r}$ for the prime factorization of n, with $p_1 < p_2 < \cdots < p_r$. The following are equivalent:*

(1) *n is a practical number;*
(2) *For all natural numbers $m \leq \sigma(n)$, we may write m as a sum of distinct divisors of n;*
(3) *For $1 \leq i \leq r$, we have*

$$p_i \leq \sigma\left(\prod_{j<i} p_i^{a_i}\right) + 1.$$

We record a trivial corollary of Theorem 1.1 for later use.

Corollary 1.2. *If $n = p_1^{a_1} \dots p_r^{a_r}$ is a practical number with $r \geq 1$, then $p_1 = 2$.*

Proof. Letting $i = 1$ in the equation embedded in (1.1) above, we obtain $p_1 \leq \sigma(1) + 1 = 2$, so $p_1 = 2$. □

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2. MINIMALIST PRACTICAL NUMBERS

If a mathematician determines that a structure or decomposition exists, he or she will then reflexively ask "Is this decomposition unique?" In our case, Theorem 1.1 tells us that if n is practical and $m < n$, then m can be decomposed as a sum of divisors of n. We are led inevitably the following definition.

Definition 2.1. We say that a practical number n is minimalist if for all natural numbers $m < n$, we may write m as a sum of distinct divisors of n in a unique way.

We are ready to state our main result.

Theorem 2.2. *Let n be a natural number. The following are equivalent:*

(1) *n is a minimalist practical number;*
(2) *For all natural numbers $m \leq \sigma(n)$, we may write m as a sum of distinct divisors of n in a unique way;*
(3) *We have $n = 2^a$ for some $a \in \mathbb{Z}_{\geq 0}$.*

Proof. The implication (3) $\implies$ (2) follows from the binary expansion for m, and the implication (2) $\implies$ (1) is immediate. It suffices to prove (1) $\implies$ (3).

Let n be a minimalist practical number, and write $n = p_1^{a_1} \dots p_r^{a_r}$ for the prime factorization of n, with $p_1 < p_2 < \cdots < p_r$. By Corollary 1.2, if $r \in \{0, 1\}$ then n is a power of 2 as desired. We suppose by way of contradiction that $r > 1$. If $p_2 < 1 + \sigma(2^{a_1})$, then the binary expansion of p_2 includes no powers of 2 greater than 2^{a_1}, and the term p_2 may be decomposed in two different ways as a sum of distinct divisors of n. So by Theorem 1.1, we conclude

$$p_2 = \sigma(2^{a_1}) + 1 = 2^{a_1+1} - 1 + 1 = 2^{a_1+1}.$$

But p_2 is a prime, and we obtain a contradiction. □

Remark. Carl Pomerance suggested an alternate proof of the implication (1) $\implies$ (3) in private correspondence. An easy inductive argument shows that the set $U = \{ 2^a : a \in \mathbb{Z}_{\geq 0} \}$ is the only collection of natural numbers such that every $m \in \mathbb{Z}_{>0}$ can be written uniquely as a sum of distinct elements of U. Thus if n is a minimalist practical number, every divisor of n must an element of U, and we conclude n is a power of 2 as desired. Notably, this argument does not depend on Stewart's characterization of the practical numbers.

Although we have characterized minimalist practical numbers, many subtler questions remain regarding the decomposition of m as a sum of divisors of n. We highlight a pair of such questions that we are especially interested in seeing answered. Let n be any practical number (or indeed any natural number), and let $m < \sigma(n)$. Under what conditions can m be written uniquely as a sum of distinct divisors of n? For $n \leq X$ a real number, what asymptotics govern the proportion of integers less than $\sigma(n)$ (or less than n) that can be written uniquely as a sum of distinct divisors of n, as $X \to \infty$?

References

[1] Giuseppe Melfi, *On two conjectures about practical numbers*, Journal of Number Theory **56** (1996), no. 1, 205–210.
[2] C. Pomerance and A. Weingartner, *On primes and practical numbers*, Ramanujan Journal **57** (2021), no. 3, 981–1000.
[3] W. Sierpiński, *Sur une propriété des nombres naturels*, Annali di Matematica Pura ed Applicata **39** (1955), 69–74.
[4] S. T. Somu, T. H. S. Li, and A. Kukla, *On some results on practical numbers*, (2023).
[5] A. K. Srinivasan, *Practical numbers*, Current Science **17** (1948), 179–180, archived from the original (PDF) on 2016-03-05. MR 0027799
[6] B. M. Stewart, *Sums of distinct divisors*, American Journal of Mathematics **76** (1954), no. 4, 779–785.
[7] A. Weingartner, *Practical numbers and the distribution of divisors*, The Quarterly Journal of Mathematics **66** (2015), no. 2, 743–758.

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